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QN.→ State and Prove Taylor's Theorm with Lagrange's form of remainder.

Ans.→ Statement :- If f be a real valued function on $[a, a+h]$ Such that

(i) All the derivatives up to $(n-1)$ are Continuous in Closed interval $[a, a+h]$

(ii) $f^n(x)$ exists in open interval

$$]a, a+h[$$

then,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots$$

$$+ \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{h^n}{\lfloor n} f^n(a+\theta h)$$

where, $0 < \theta < 1$

Proof :- To Prove this theorm, let us Consider a function "F" defind by

$$F(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{\lfloor 2} f''(x) +$$

$$\cdots + \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^{(n-1)}(x)$$

$$+ \frac{A}{\lfloor n} (a+h-x)^n$$

where, A is Const. to be determind.

To, determine A, we choose A Such that,

$$F(a) = F(a+h)$$

$$\text{or, } f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}$$

$$(a) + \frac{A}{\lfloor n} \cdot h^n = f(a+h)$$

ⓐ.

Now, from hypothesis (i)

$f, f', f'' \cdots f^{n-1}$ are differentiable in $]a, a+h[$

Also, $(a+h-x)$, $\frac{(a+h-x)^2}{\lfloor 2}$, ---- $\frac{(a+h-x)^n}{\lfloor n}$

are all diff$^{=}$ in $]a, a+h[$

Hence, from above, we can so that F is also differentiable in $]a, a+h[$

$$F'(x) = \cancel{f'(x)} - \cancel{f'(x)} + (a+h-x) f''(x) - \frac{\cancel{2}(a+h-x)}{\lfloor \cancel{2}} f''(x) + \frac{(a+h-x)^2}{\lfloor 2} f'''(x) - \frac{(a+h-x)^2}{\lfloor 2} f'''(x)$$

$$+ ------ - \frac{(a+h-x)^{n-2}}{\lfloor n-2} f^{(n-1)}(x)$$

$$+ \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^{(n)}(x) - \frac{(a+h-x)^{n-1}}{\lfloor n-1} A$$

On simplification

$$F'(x) = \frac{(a+h-x)^{n-1}}{\lfloor n-1} \left[f^n(x) - A\right] \text{ --- (b)}$$

Hence, F satisfies all the condition of Rolle's theorem

$$\therefore F'(a+\theta h) = 0 \text{ --- (c)} \quad [0<\theta<1]$$

from (b), it is clear

on putting $x = a+\theta h$ in (b), we have

$$F'(a+\theta h) = \frac{(\cancel{a}+h-\cancel{a}-\theta h)^{n-1}}{\lfloor n-1} \left[f^n(a+\theta h) - A\right]$$

(d)

from (c) & (d)

$$0 = \frac{(1-\theta)^{n-1} \cdot h^{n-1}}{\lfloor n-1} \left[f^n(a+\theta h) - A\right]$$

$$\because \frac{(1-\theta)^{n-1} h^{n-1}}{\lfloor n-1} \neq 0$$

Hence, $A = f^n(a+\theta h)$

Now, Putting the value of A in eqn. (a), we have,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\underline{|2}} f''(a) + \text{-----}$$

$$+ \frac{h^{n-1}}{\underline{|n-1}} f^{(n-1)}(a) + \frac{h^n}{\underline{|n}} \times f^n(a+\theta h)$$

Hence, $(n+1)$th term

$$= \frac{h^n}{\underline{|n}} f^n(a+\theta h)$$

is Know as Lagrange's form of remainder.